Finite Simple Groups
Exercise Sheet 4
Due 21.05.2019

Exercise 1 (4 Points).
Let G be a finite group of order $2n$. Using the fact that every element has an inverse, prove that G has an element of order 2.

Exercise 2 (10 Points).
Let H and N be groups and let $\varphi : H \to \mathrm{Aut}(N)$ be a group homomorphism (*i.e.* H acts on N via automorphisms). On the set $N \times H$, define the following operation $*$:

$$(n, h) * (n', h') = (n\varphi_h(n'), hh'),$$

where φ_h denotes the automorphism $\varphi(h)$ of N.

1. Prove that $N \times H$ with the operation $*$ is a group. We denote this group by $N \rtimes_\varphi H$.
2. Show that there are two subgroups $\tilde{N}$ and $\tilde{H}$ of $N \rtimes_\varphi H$ which are isomorphic to N and H respectively and satisfy the following properties: $N \rtimes_\varphi H = \tilde{N} * \tilde{H}$, $\tilde{N}$ is normal in $N \rtimes_\varphi H$ and $\tilde{N} \cap \tilde{H} = \{1\}$.

Suppose now that G is an arbitrary group and let H and N be two subgroups of G such that $G = NH$, N is normal in G and $H \cap N = \{1\}$.

3. Show that for every element g of G there are unique elements n in N and h in H such that $g = nh$.
4. Deduce that there exists a group homomorphism $\varphi : H \to \mathrm{Aut}(N)$ such that that $G \cong N \rtimes_\varphi H$.

In the situation above, we say that G is the *semidirect product* of N and H and we denote it by $G = N \rtimes H$, without specifying the action via automorphisms.

5. Characterise when the direct product and the semidirect product are equal.

Exercise 3 (6 Points).
Let $T = \langle t \rangle$ be a cyclic group of order 2 and consider an abelian group A. Notice that there exists a group homomorphism $\varphi : T \to \mathrm{Aut}(A)$ given by the map $t \mapsto \varphi_t$, where $\varphi_t(a) = a^{-1}$ for a in A. Let $G = A \rtimes T$ be with respect to this group homomorphism, and identify (as we have seen in Exercise 2) the groups A and T with the corresponding subgroups of G of the semidirect product.

1. Deduce that $|G| = 2|A|$ and that every element of $G \setminus A$ has order 2.

Suppose now that A is cyclic of order n. Then G is a *dihedral group* of order $2n$.

2. Suppose that $n \geq 3$ and let a be a generator of A. Prove that the center of G consists of the identity and all elements of the form a^k with k satisfying the equation $2k = n$.
3. For $n = 2$, show that $G \cong C_2 \times C_2$.